VECTORS

CLASS XI

PHYSICS

Total number of Questions in **Vectors** are :

In Chapter Examples 24

Solved Examples 17

Total no. of questions 41

1. VECTOR

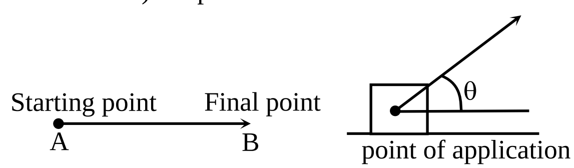
(a) The physical quantities which have magnitude and direction and which can be added according to the triangle rule, are called vector quantities. Examples : force, linear momentum, electric field, magnetic field etc.

(b) If a physical quantity has magnitude as well as direction but does not add up according to the triangle rule, it will not be called a vector quantity. Example Electric current in a wire has both magnitude and direction but there is no any meaning of triangle rule. Thus, electric current is not a vector quantity.

1.1 Various types of vectors.

(a) Polar Vectors :

Vectors having starting point (as in case of displacement) or point of application (as in case of force) are polar vectors.



(b) Axial vectors

Vectors representing rotational effects and are always along the axis of rotation in accordance with right hand screw rule are axial vectors.

Ex. Angular displacement (θ); Angular velocity (ω); Angular acceleration (α) ; Torque (τ) etc.

1.2 Some other types of vectors :

(a) Zero vector :

A vector with zero magnitude is called zero vector.

(b) Proper vector :

A vector with non-zero magnitude is called proper vector.

(c) Like vectors:

Two (or more) vector are called like vectors if their supports are same or parallel and are in the same sense.

(d) Unlike vectors :

Two vectors are called unlike vectors if their supports are same or parallel and are in the opposite sense.

(e) Equal vectors :

Two vectors are called equal (or equivalent) vectors if they have equal magnitude, same or parallel supports and same sense.

(f) Collinear vectors :

Two (or more) vectors are called collinear vectors if they have same or parallel supports.

(g) Coplanar vectors :

Three (or more) vectors are called coplanar vectors if they lie in the same plane or are parallel to the same plane. Two (free) vectors are always coplanar.

(h) Negative vector :

A vector having the same magnitude as that of the given vector but directed in the opposite sense is called the negative of the given vector.

(i) Unit vector :

A vector with magnitude of unity is called unit vector. Unit vector in the direction of a is

$$\hat{a} = \mathbf{a}/|a|$$

Example based on Types of Vectors

Ex.1 Give an example of physical quantity which-
(A) has neither unit nor direction
(B) has direction but not a vector
(C) can be either a scalar or a vector
(D) is neither a scalar nor a vector

Sol. (A) refractive index; strain
(B) current
(C) angular displacement
(D) moment of inertia

Ex.2 Can the resultant be zero in case of –
(a) two unequal vector
(b) three coplanar vectors
(c) three non-coplanar vectors

Sol. (a) No (b) Yes (c) No

Ex.3 Discuss whether angular displacement is a vector or not –

Sol. Angular displacement represents rotational effect, so appears to be an axial vector. Also, if obeys the law of parallelogram of addition but it has been found that for large angular displacement it does not obey law of commutation.

Ex.4 Does it make sense to call a quantity vector when its magnitude is zero. What such vectors are called and why needed ?

Sol. Consider a vector **A** whose length is reduced to zero by coinciding the initial and terminal points. Hence modulus of the vector is zero and its direction indeterminate is it may posses any direction such vectors are called zero or null vectors. It is needed because, the sum of two equal and opposite vectors is a null vector or area of sphere in vector form is a null vector.

Ex.5 We can order events in time and there is a sense of time, distinguishing past, present and future is therefore, time a vector-

Sol. Time always flow from past to present and then the future, so a direction can be assigned to time. However as the direction is unique, it is not to be stated i.e. specified. Hence it is not a vector quantity.

Ex.6 Explain why current is not a vector although it appears to posses a direction –

Sol. Current is not a vector as :

(i) The direction associated with current is not a true direction but it merely indicates the sense of charge flow.

(ii) Current also does not obey the law of parallelogram of addition.

(iii) Current is also defined as scalar product of current density (**J**) with area **ds** is-

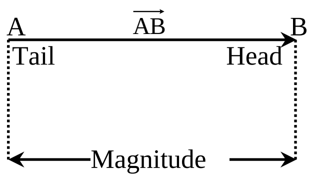
$$I = \int \mathbf{J} \cdot \mathbf{ds}$$

2. REPRESENTATION OF VECTOR

There are two methods for representation of vectors.

(a) Graphical method.

(b) Mathematical method.



(a) Graphical method :

The length of the arrow shows the magnitude and head of the arrow shows the direction.

(b) Mathematical Representation :

(i) In the form of components : If a_x is a component of any vector in x-direction, a_y is a component in y-direction and a_z is component in z-direction then

$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

where a_x, a_y, a_z may be the co-ordinates of point a.

$$|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

(ii) Unit vector method : If we want to represent any vector in mathematical form then we will multiply the magnitude of that vector with unit vector of the direction.

if a = magnitude of vector

and $\hat{a}$ = unit vector

then $\vec{a} = a\,\hat{a}$

Example based on Representation

Ex.7 If a particle moves 5m in + x-direction. Show the displacement of the particle-

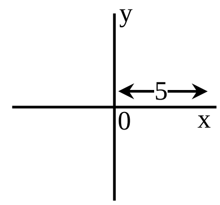
(A) $5\,\hat{j}$ (B) $5\,\hat{i}$

(C) $-5\,\hat{j}$ (D) $5\,\hat{k}$

Sol. Magnitude of vector = 5

Unit vector in + x direction is $\hat{i}$

displacement = $5\,\hat{i}$

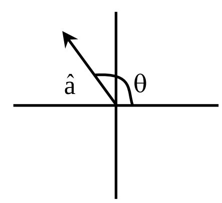


Hence correct answer is (B).

Note : Remember the unit vector in x, y plane of a vector which makes angle θ with x-axis.

$$\hat{a} = \hat{i}\cos\theta + \hat{j}\sin\theta$$

(iii) Co-ordinate Method : If point A is (x_1, y_1, z_1) and point B is (x_2, y_2, z_2) then find out $\overrightarrow{AB}$

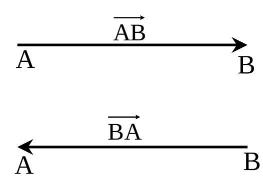


$$\vec{B} - \vec{A} = (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$$

So $\overrightarrow{AB} = (x_2 - x_1)\,\hat{i} + (y_2 - y_1)\,\hat{j} + (z_2 - z_1)\,\hat{k}$

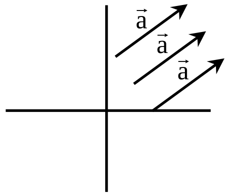
Important Points :

(i) If we multiply a vector with negative sign, then the magnitude will not change only direction will change



$$\overrightarrow{BA} = -\overrightarrow{AB}$$

(ii) Two vectors are called equal if their magnitudes and direction are same. So we can transfer any vector (to any where) parallel to itself.



(iii) If two vectors has same direction then the unit vector of both of them will be similar. Because magnitude of unit vector is unity.

3. VECTOR ADDITION

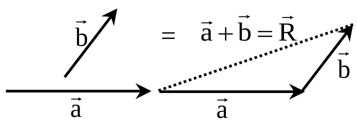
There are two methods for addition of vectors :

3.1 Graphical method
3.2 Mathematical method

3.1 Graphical Method :

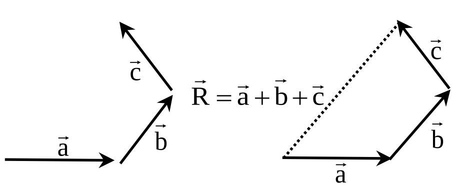
(i) Triangle rule (Used to add two vectors only) :

If $\vec{a}$ and $\vec{b}$ are the two vectors to be added, a diagram is drawn in which the tail of $\vec{b}$ coincides with the head of $\vec{a}$. The vector joining the tail of $\vec{a}$ with the head of $\vec{b}$ is the **vector sum** of $\vec{a}$ and $\vec{b}$. Figure shows the construction.



(ii) Polygon method : (used to add more than two vectors)

We use this method for more than two vector. Suppose $\vec{a}, \vec{b}, \vec{c}$ are three vectors to be added. A diagram is drawn in which the tail of $\vec{b}$ coincides with the head of $\vec{a}$ and tail of $\vec{c}$ coincides with head of $\vec{b}$. The vector joining the tail of $\vec{a}$ and head of $\vec{c}$ is called the **resultant vector** and this is the vector sum of three given vectors $(\vec{a}+\vec{b}+\vec{c})$

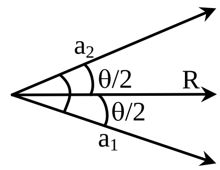


Result : 1.

If three or more vectors themselves complete a triangle or a polygon then their sum-vector cannot be drawn. It means that the sum of these vectors is zero.

Result : 2.

If there are two vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ with equal magnitude, then the resultant of their addition will bisect the angle between them.



Result : 3.

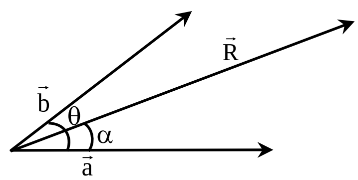
If we add two different vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ with equal magnitude and angle between them is 120º, then the resultant would bisect the angle and magnitude would be equal to each of the magnitude of vector.

3.2 Mathematical method :

3.2.1 For two vectors :

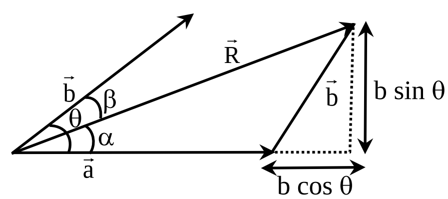
If two vectors $\vec{a}$ and $\vec{b}$ makes angle θ to each other then the magnitude of their vector addition

$$R = |\vec{a}+\vec{b}| = \sqrt{(a^2+b^2+2ab\cos\theta)}$$



and if the resultant vector makes the angle α with the vector $\vec{a}$ then it is given by

$$\tan\alpha = \left(\frac{b\sin\theta}{a+b\cos\theta}\right)$$



and if the resultant vector makes the angle β with the vector $\vec{b}$ then it is given by

$$\tan\beta = \left(\frac{a\sin\theta}{b+a\cos\theta}\right)$$

and $\quad R^2 = (a + b\cos\theta)^2 + (b\sin\theta)^2$

Where θ is the angle between a and b

Important Results :

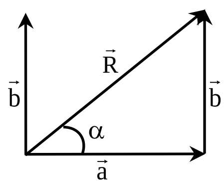
(a) If $\quad \theta = 0 \Rightarrow \vec{a} \,||\, \vec{b}$

then $\qquad R = |\vec{a}| + |\vec{b}| = \vec{R}_{max}$.

(b) If $\qquad \theta = \pi \Rightarrow \vec{a}$ anti || $\vec{b}$

then $\qquad R = |\vec{a}| - |\vec{b}| = \vec{R}_{min}$

(c) If $\theta = \dfrac{\pi}{2} \Rightarrow \vec{a} \perp \vec{b}$, $R = \sqrt{a^2 + b^2}$



Here $\tan \alpha = \left(\dfrac{b}{a}\right)$

(d) $\qquad |\vec{a}| = |\vec{b}| = a$

$$|\vec{R}| = 2a \cos \frac{\theta}{2}$$

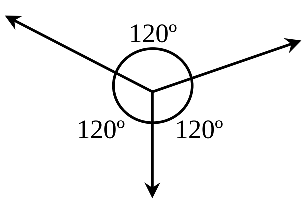
and $\qquad \tan \alpha = \tan \dfrac{\theta}{2}$

$\therefore \qquad \alpha = \dfrac{\theta}{2}$

(e) If $|\vec{a}| = |\vec{b}|$ and $\theta = 120^\circ$ then $|\vec{R}| = a$

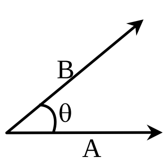
(f) If three vectors of equal magnitudes makes an angle of 120º with each other then the resultant vector will be zero.

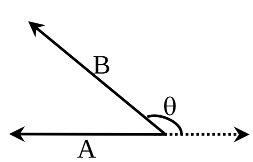
$$R = 0$$

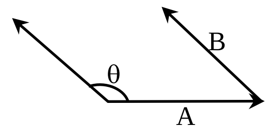


(g) If n vectors of equal magnitude makes the angle of equal measure with each other then the resultant vector will be zero.

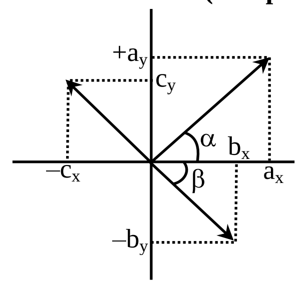
(h) Angle between two vectors means smaller of the two angles between the vectors when they are placed tail to tail by displacing either of the vector parallel to itself (i.e. $0 \le \theta \le \pi$).







3.2.2 For more then two vectors (Component method) :



$$\vec{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$$

$$\vec{b} = b_x \hat{i} + b_y \hat{j} + b_z \hat{k}$$

$$\vec{c} = c_x \hat{i} + c_y \hat{j} + c_z \hat{k}$$

and $\quad R = \vec{a} + \vec{b} + \vec{c}$

$$= (a_x + b_x + c_x)\hat{i} + (a_y + b_y + c_y)\hat{j} + (a_z + b_z + c_z)\hat{k}$$

Suppose $\qquad a_z = b_z = c_z = 0$

and $\qquad \vec{a} = a_x \hat{i} + a_y \hat{j}$

$$\vec{b} = b_x \hat{i} - b_y \hat{j}$$

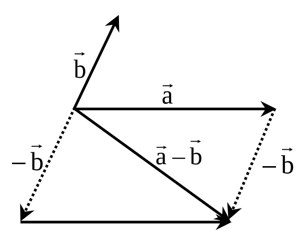
$$\vec{c} = -c_x \hat{i} + c_y \hat{j}$$

So the resultant $\qquad \vec{R} = R_x \hat{i} + R_y \hat{j}$

$$\vec{R} = (a_x + b_x - c_x)\hat{i} + (a_y - b_y + c_y)\hat{j}$$

4. SUBTRACTION OF VECTORS

Let $\vec{a}$ and $\vec{b}$ be two vectors. We define $\vec{a} - \vec{b}$ as the sum of the vector $\vec{a}$ and the vector $(-\vec{b})$. To subtract $\vec{b}$ from $\vec{a}$, invert the direction of $\vec{b}$ and add to $\vec{a}$. Figure shows the process.



Example based on Addition & Subtraction of Vectors

Ex.8 Under what condition the sum and difference of two vectors will be equal in magnitude.

Sol. When the two vectors are equal in magnitude and perpendicular to each other

$$|\mathbf{A} + \mathbf{B}| = |\mathbf{A} - \mathbf{B}|$$

Squaring both the side

$A^2 + B^2 + 2\mathbf{A}.\mathbf{B} = A^2 + B^2 - 2\mathbf{A}.\mathbf{B}$

$4\mathbf{A}.\mathbf{B} = 0$

$\mathbf{A}.\mathbf{B} = 0$

$\mathbf{A} \perp \mathbf{B}$

Ex.9 There are two displacement vectors, one of magnitude 3m and other of 4m. How should the two vectors be added so that the resultant vector be : (a) 7m (b) 1m (c) 5m.

Sol. (a) For 7m both the vector should be parallel i.e. angle between them should be zero

(b) For 1m both the vectors should be anti parallel i.e. angle between them should be 180º

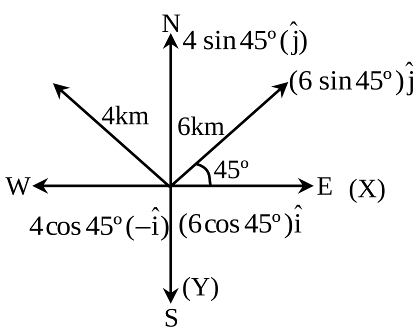
(c) For 5m both the vectors should be perpendicular to each other i.e. angle between them should be 90º

Ex.10 A car travels 6 km towards north at an angle of 45º to the east and then travels distance of 4 km towards north at an angle of 135° to the east. How far is its final position due east and due north? How far is the point from the starting point? What angle does the straight line joining its initial and final position makes with the east? What is the total distance travelled by the car ?

Sol. Net movement along X – direction

$= (6 - 4)\cos 45°\,\hat{i}$

$= 2 \times \frac{1}{\sqrt{2}} = \sqrt{2}$ km



Net movement along Y – direction

$= (6 + 4)\sin 45°\,\hat{j}$

$= 10 \times \frac{1}{\sqrt{2}} = 5\sqrt{2}$ km

Net movement form starting point

$= 6 + 4 = 10$ km

Angle which makes with the east direction

$$\tan\theta = \frac{\text{Y} - \text{component}}{\text{X} - \text{component}}$$

$$= \frac{5\sqrt{2}}{\sqrt{2}}$$

$\theta = \tan^{-1}(5)$

$\theta = 79º$

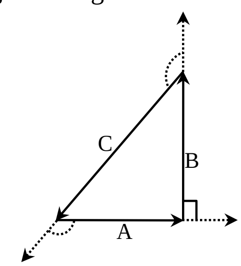
Total distance travelled = 10 km

Ex.11 Given that $\mathbf{A} + \mathbf{B} + \mathbf{C} = 0$, but of three vectors two are equal in magnitude and the magnitude of third vector is $\sqrt{2}$ times that of either of the two having equal magnitude. Then the angles between vectors are given by -

(A) 30º, 60º, 90º (B) 45º, 45º, 90º

(C) 45º, 60º, 90º (D) 90º, 135º, 135º

Sol.(D) From polygon law, three vectors having summation zero should form a closed polygon (triangle). Since the two vectors are having same magnitude and the third vector is $\sqrt{2}$ times that of either of two having equal magnitude. i.e. the triangle should be right angled triangle.



Angle between **A** and **B** is 90º,

Angle between **B** and **C** is 135º

Angle between **A** and **C** is 135º

Ex.12 If $\mathbf{A} = 4\hat{i} - 3\hat{j}$ and $\mathbf{B} = 6\hat{i} + 8\hat{j}$ obtain the scalar magnitude and directions of **A**, **B**, **A** + **B**; **A** – **B** and **B** – **A**,

Sol. Magnitude of **A** = $\sqrt{(4)^2 + (-3)^2} = 5$

$\tan\theta = -\frac{3}{4} \Rightarrow \theta = \tan^{-1}\left(\frac{3}{4}\right)$

Magnitude of **B** = $\sqrt{6^2 + 8^2} = 10$

$\tan\theta = \frac{8}{6} = \frac{4}{3} \Rightarrow \theta = \tan^{-1}\left(\frac{4}{3}\right) = 53º$

$\mathbf{A} + \mathbf{B} = 4\hat{i} - 3\hat{j} + 6\hat{i} + 8\hat{j}$

$= 10\hat{i} + 5\hat{j}$

$(\mathbf{A} + \mathbf{B}) = \sqrt{(10)^2 + (5)^2} = 11.2$

$\tan\theta = \frac{5}{10} = \frac{1}{2}$

$\theta = \tan^{-1}$ (0.51 = 26.5º approx)

$\mathbf{A} - \mathbf{B} = 4\hat{i} - 3\hat{j} + (6\hat{i} + 8\hat{j})$

$= 4\hat{i} - 6\hat{i} - 3\hat{j} - 8\hat{j}$

$= -2\hat{i} - 11\hat{j}$

$B - A = 6\hat{i} + 8\hat{j} - 4\hat{i} + 3\hat{j} = 2\hat{i} + 11\hat{j}$

$|B - A| = \sqrt{4 + 121} = \sqrt{125} = 5\sqrt{5}$

$\tan\theta = \frac{11}{2} \Rightarrow \theta = \tan^{-1}\left(\frac{11}{2}\right)$

Ex.13 Prove that the vectors

$\mathbf{A} = 2\hat{i} + \hat{j} - 2\hat{k}$;

$\mathbf{B} = -\hat{i} + 3\hat{j} + 4\hat{k}$ and $\mathbf{C} = 4\hat{i} - 2\hat{j} - 6\hat{k}$

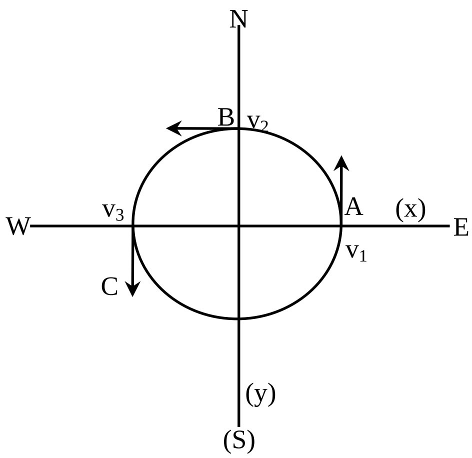
can form a triangle.

Sol. Let us add the vector **B** and **C**

$\mathbf{B} + \mathbf{C} = (-\hat{i} + 3\hat{j} + 4\hat{k}) + (4\hat{i} - 2\hat{j} - 6\hat{k})$

$= 3\hat{i} + \hat{j} - 2\hat{k} = \mathbf{A}$

Ex.14 A body is moving with uniform speed v on a horizontal circle in anticlockwise direction from A as shown in figure. What is the change in velocity in (a) half revolution (b) first quarter revolution



Sol. Change in velocity in half revolution

$\Delta\mathbf{v} = \mathbf{v}_C - \mathbf{v}_A$

$= v(-\hat{j}) - v(\hat{j})$

$\Delta\mathbf{v} = -2v\hat{j}$

$|\Delta\mathbf{v}| = 2v$ directed towards negative y-axis

change in velocity in first quarter revolution

$\Delta\mathbf{v} = \mathbf{v}_B - \mathbf{v}_A$

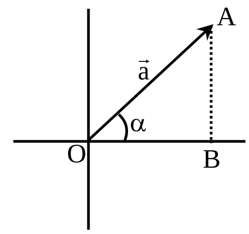
$= v(-\hat{i}) - v(\hat{j})$

$= -v(\hat{i} + \hat{j})$

$|\Delta\mathbf{v}| = \sqrt{2}\,v$ and directed towards south-west.

5. RESOLUTION OF VECTORS

(a) See the figure below. Vector $\vec{a}$ is in x–y plane and it makes the angle α with the x–axis.



(b) $\overrightarrow{OB}$ and $\overrightarrow{BA}$ are the projection of vector on horizontal and vertical axis respectively.

(c) Applying trigonometric theory we can easily find out that OB = A cos α and BA = A sin α.

(d) So $\overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{BA}$

$\Rightarrow \overrightarrow{OA} = \vec{A} = A_x\hat{i} + A_y\hat{j}$

$= A\cos\alpha\,\hat{i} + A\sin\alpha\,\hat{j}$

Example based on **Resolution**

Ex.15 A force of (10.5 N) acts on a particle along a direction making an angle of 37º will the vertical, find the component of the force in the vertical direction.

(A) 7.3 N (B) 8 N

(C) 8.4 N (D) 6 N

Sol The component of the force in the vertical direction will be

$F_1 = F\cos\theta = (10.5)\text{ N}\cos 37^\circ$

$= (10.5\text{ N}) = 8.40\text{ N}$

Hence correct answer is (C).

6. MULTIPLICATION OF TWO VECTORS

Two types of multiplication

6.1 Scalar (or dot) product of two vectors.

6.2 Vector (or cross) product of two vectors.

6.1 Scalar (or dot) product of two vectors :

The scalar or dot product of two vectors **a** and **b** denoted by **a**. **b** (read as **a** dot **b**) its defined as

a . **b** = |**a**| |**b**| cos θ = ab cos θ

where a = |**a**| ; b = |**b**| and θ is the angle between **a** and **b**

6.1.1 Remarks :

(1) If **a** = 0 or **b** = 0 we define **a** . **b** = 0 (as θ is meaningless)

(2) The dot product of two vectors is a scalar quantity

(3) If θ = 0 (i.e. **a** and **b** are like vectors) **a** . **b** = ab (as cos 0 = 1)

(4) If θ = π (i.e. **a** and **b** are unlike vectors) **a** . **b** = – ab (as cos π = –1)

(5) Condition for two vectors to be perpendicular;

If a and b are perpendicular , the angle between them is $\pi/2$ and we obtain

$\mathbf{a}.\ \mathbf{b} = ab \cos \pi/2 = 0$

Conversely, if $\mathbf{a}.\ \mathbf{b} = 0$ i.e. if $ab \cos \theta = 0$ then either $a = 0$ or $b = 0$ or $\cos \theta = 0$ it follows that either (or both) of the vectors is a zero or else they are perpendicular.

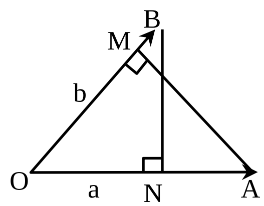
(6) Note that $\mathbf{a}.\mathbf{b} > 0$ if $0 \le \theta < \pi/2$ i.e. if angle between vectors in acute.

$\mathbf{a}.\mathbf{b} = 0$ if $\theta = \pi/2$ i.e. if a and b are perpendicular.

$\mathbf{a}.\mathbf{b} < 0$ if $\pi/2 < \theta \le \pi$ if angle between vectors is obtuse

6.1.2 Geometrical interpretation of scalar product

Let **OA** and **OB** represent vectors **a** and **b** respectively. Then



$a = |\mathbf{a}| = |\mathbf{OA}| = OA$

$b = |\mathbf{b}| = |\mathbf{OB}| = OB$

Let M, N be the feet of the perpendiculars from A, B on OB, OA respectively

Then magnitude of projection of **a** on **b**

$= OM = OA \cos \theta$ ($\therefore \cos \theta = OM/OA$ in ΔOMA)

$= a \cos \theta$

$\therefore\ \mathbf{a}.\mathbf{b} = ab \cos \theta = b(a \cos \theta) = b.$ (projection of **a** on **b**)

similarly magnitude of projection of **b** on **a**.

$= ON = OB \cos \theta$ ($\therefore \cos \theta = ON/OB$ in ΔONB)

$= b \cos \theta$

$\mathbf{a}\ .\ \mathbf{b} = ab \cos \theta = a(b \cos \theta) = a.$ projection of **b** on **a**. Thus **a**. **b** can be defined as the product of the modulus of one vector and the projection of the other vector upon it.

6.1.3 Remarks :

(1) Given two vectors **a** and **b**, the projection of one vector on another can be found by using the formula.

projection of **a** on **b** = $\mathbf{a} \cos \theta = \dfrac{a.b}{|b|}$

and projection of **b** on **a** = $\mathbf{b} \cos \theta = \dfrac{a.b}{|a|}$

(ii) Angle between two vectors :

Angle between two vectors **a** and **b** can be found by using

$\mathbf{a}\ .\ \mathbf{b} = ab \cos \theta \Rightarrow \cos \theta = \dfrac{a.b}{ab} = \dfrac{a.b}{|a||b|}$

(iii) Square of a vector

The scalar product of a vector **a** with itself is called the square of the vector **a**, and is written as $\mathbf{a}.\ \mathbf{a}$ or $(a)^2$

$(a)^2 = \mathbf{a}\ .\ \mathbf{a} = aa \cos 0 = a\ .a.\ 1 = a^2 = |\mathbf{a}|^2$

The magnitude of a vector can be found by using

$|\mathbf{a}| = \sqrt{a.a}$ $\quad$ ($\therefore\ |\mathbf{a}|^2 = \mathbf{a}\ .\ \mathbf{a}$)

6.1.4 Squares and scalar products of $\hat{i}; \hat{j}; \hat{k}$

Since $\hat{i}, \hat{j}$ and $\hat{k}$ are unit vectors along the co-ordinate axes i.e. along three mutually perpendicular lines, we have :

$\hat{i}.\hat{i} = 1.1 \cos 0 = 1$ similarly $\hat{j}.\hat{j} = 1$; $\hat{k}.\hat{k} = 1$

Also, $\hat{i}.\hat{j} = 1.1\cos 90^\circ = 1.1.0 = 0$,

similarly $\hat{j}.\hat{i} = 0$; $\hat{j}.\hat{k} = 0$

$\hat{k}.\hat{j} = 0$; $\hat{k}.\hat{i} = 0$, $\hat{i}.\hat{k} = 0$

6.1.5 Properties of scalar (or dot product)

(1) The scalar product is commutative

i.e. $\mathbf{a}.\ \mathbf{b} = \mathbf{b}.\mathbf{a}$

(2) If **a**, **b** are any vectors and m is any real number (scalar) then

$(m\mathbf{a})\ .\ \mathbf{b} = m(\mathbf{a}.\ \mathbf{b}) = \mathbf{a}\ .\ (m\ \mathbf{b})$

Corollary 1 : $\mathbf{a}\ (-\mathbf{b}) = -(\mathbf{a}\ .\ \mathbf{b}) = (-\mathbf{a})\ .\ (\mathbf{b})$

Corollary 2 : $(-\mathbf{a})\ .\ (-\mathbf{b}) = \mathbf{a}.\ \mathbf{b}$

6.1.6 Scalar product of two vectors in terms of their rectangular components.

Let $\mathbf{a} = a_1\ \hat{i} + a_2\ \hat{j} + a_3\hat{k}$

and $\mathbf{b} = b_1\ \hat{i} + b_2\ \hat{j} + b_3\hat{k}$

then $\mathbf{a}\ .\ \mathbf{b} = (a_1\hat{i} + a_2\ \hat{j} + a_3\hat{k}).(b_1\ \hat{i} + b_2\ \hat{j} + b_3\hat{k})$

$= a_1b_1 + a_2b_2 + a_3b_3$

Corollary 1: $\mathbf{a}.\ \mathbf{b} = 0$

If $a_1\ b_1 + a_2\ b_2 + a_3\ b_3 = 0$

Thus if $\mathbf{a} \perp \mathbf{b}$ or if $a_1\ b_1 + a_2\ b_2 + a_3\ b_3 = 0$

Corollary 2 : $\mathbf{a} \parallel \mathbf{b}$ (collinear)

if $a_1/b_1 = a_2/b_2 = a_3/b_3$

6.1.7 Angle between two vectors

$\cos \theta = \mathbf{a}\ .\ \mathbf{b}\ /\ ab$

Let $\mathbf{a} = a_1\hat{i} + a_2\ \hat{j} + a_3\hat{k}$

and $\mathbf{b} = b_1\ \hat{i} + b_2\ \hat{j} + b_3\hat{k}$

Then **a**. **b** = $a_1b_1 + a_2b_2 + a_3b_3$

a = |**a**| = $\sqrt{a_1^2+a_2^2+a_3^2}$: |**b**| = b = $\sqrt{b_1^2+b_2^2+b_3^2}$

$$\cos\theta = \frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}$$

$$\theta = \cos^{-1}\left[\frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}\right]$$

6.1.8 Examples of dot product :

(i) Work (**W**) is the dot product of force (**F**) and displacement (**r**)

W = **F** . **r**

(ii) Power (**P**) is the dot product of force (**F**) and velocity (**v**).

P = **F** . **v**

(iii) Electric flux (ϕ) is the dot product of intensity of electric field (**E**) and normal area **A**.

ϕ = **E** . **A**

Example based on Scalar Product

Ex.16 Let **a** = $2\hat{i} + 3\hat{j} - \hat{k}$; **b** = $-\hat{i} + 3\hat{j} + 4\hat{k}$. Evaluate

(i) |**a**| ; |**b**|

(ii) **a**. **b**

(iii) the angle between the vectors **a** and **b**

(iv) the projection of **a** on **b**

(v) the projection of **b** on **a**

(vi) area of the ΔAOB where O is origin

Sol. Given **a** = $2\hat{i} + 3\hat{j} - \hat{k}$, **b** = $-\hat{i} + 3\hat{j} + 4\hat{k}$

(i) |**a**| = $\sqrt{2^2+3^3+(-1)^2} = \sqrt{4+9+1} = \sqrt{14}$

|**b**| = $\sqrt{(-1)^2+3^2+4^2} = \sqrt{1+9+16} = \sqrt{26}$

(ii) **a**. **b** = 2 (–1) + 3 × 3 + (–1) (4) = 3

(iii) The angle θ between the vectors **a** and **b** is given by

$$\cos\theta = \frac{a.b}{|a||b|} = \frac{3}{\sqrt{14}\sqrt{26}} = \frac{3}{2\sqrt{91}}$$

(iv) The projection of **a** on **b** = |**a**| cos θ

$$= \sqrt{14}\times\frac{3}{\sqrt{14}\sqrt{26}} = \frac{3}{\sqrt{26}}$$

(v) The projection of **b** on **a** = |**b**| cos θ

$$= \sqrt{26}\times\frac{3}{\sqrt{14}\sqrt{26}} = \frac{3}{\sqrt{14}}$$

(vi) Area of ΔAOB = $\frac{1}{2}$ |**a**| |**b**| sin θ

Now $\sin^2\theta = 1 - \cos^2\theta = 1 - \left(\frac{3}{2\sqrt{91}}\right)^2$

$$= 1 - \frac{9}{364} = \frac{355}{364}$$

Area of ΔAOB = $\frac{1}{2}\sqrt{14}\ \sqrt{26}.\sqrt{\frac{355}{364}}$

$$= \sqrt{\frac{355}{364}}\sqrt{355} = 9.42 \text{ sq. units approx.}$$

Ex.17 Prove that the three vectors $3\hat{i}+\hat{j}+2\hat{k}$, $\hat{i}-\hat{j}-\hat{k}$ and $\hat{i}+5\hat{j}-4\hat{k}$ are at right angles to one another.

Sol. Let **a** = $3\hat{i}+\hat{j}+2\hat{k}$ **b** = $\hat{i}-\hat{j}-\hat{k}$ we note that all the three vectors are non-zero

a . **b** = $(3\hat{i}+\hat{j}+2\hat{k}).(\hat{i}-\hat{j}-\hat{k})$

= 3(1) + 1(–1) + 2(–1) = 0

Thus the dot product of two non-zero vectors **a** and **b** is zero, therefore these vectors are perpendicular to each other.

Again **b** . **c** = $(\hat{i}-\hat{j}-\hat{k}).(\hat{i}+5\hat{j}-4\hat{k})$

= (1) (1) + (–1) (5) + (–1) (–4)

= 0

and **c** . **a** = $(\hat{i}+5\hat{j}-4\hat{k}).(3\hat{i}+\hat{j}+2\hat{k})$

= (1) (3) + 5(1) + (–4) (2) = 0

As above, it follows that **b** , **c** are perpendicular and **c**, **a** are perpendicular. Hence all the three given vectors are perpendicular to each other.

Ex.18 Find the value of λ so that the two vectors $2\hat{i}+3\hat{j}-\hat{k}$ and $-4\hat{i}-6\hat{j}-\lambda\hat{k}$ are

(i) parallel (ii) perpendicular to each other

Sol. Let **a** = $2\hat{i} + 3\hat{j} - \hat{k}$ and **b** = $-4\hat{i} - 6\hat{j} + \lambda\hat{k}$

(i) **a** and **b** are parallel to each other

$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$ i.e. if $\frac{2}{-4} = \frac{3}{-6} = \frac{-1}{\lambda}$

⇒ λ = 2

(ii) **a** and **b** are perpendicular to each other if **a**. **b** = 0

i.e. if 2(–4) + 3(–6) + (–1) (λ) = 0

λ = – 8 – 18 = –26

Ex.19 Show that the vectors **a** = $3\hat{i} - 2\hat{j} + \hat{k}$, **b** = $\hat{i} - 3\hat{j} + 5\hat{k}$, **c** = $2\hat{i} + \hat{j} - 4\hat{k}$ form a right angled triangle

Sol. We have **b** + **c** = $(\hat{i} - 3\hat{j} + 5\hat{k}) + (2\hat{i} + \hat{j} - 4\hat{k})$

$= 3\hat{i} - 2\hat{j} + \hat{k} = \mathbf{a}$

$\Rightarrow$ **a**, **b**, **c** are coplanar

Hence no two of these vectors are parallel, therefore, the given vectors form a triangle.

$\mathbf{a} . \mathbf{c} = (3\hat{i} - 2\hat{j} + \hat{k}) . (2\hat{i} + \hat{j} - 4\hat{k})$

$= (3)(2) + (-2)(1) + (1)(-4) = 0$

Hence the given vectors form a right angled triangle.

Ex.20 A particle, under constant force $\hat{i} + \hat{j} - 2\hat{k}$ gets displaced from point A(2, –1, 3) to B(4, 3, 2). Find the work done by the force-

Sol Force = $\hat{i} + \hat{j} - 2\hat{k}$

displacement = $\mathbf{d} = \overrightarrow{AB} = (4\hat{i} + 3\hat{j} + 2\hat{k})$

$- (2\hat{i} - \hat{j} + 3\hat{k}) = (2\hat{i} + 4\hat{j} - \hat{k})$

Work done

$= \mathbf{F} . \mathbf{d} = (\hat{i} + \hat{j} - 2\hat{k}) . (2\hat{i} + 4\hat{j} - \hat{k})$

(1) (2) + (1) (4) + (–2) (–1) = 2 + 4 + 2 = 8 units

6.2 Cross product or vector product of two vectors

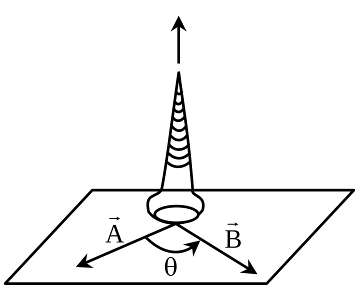
The vector product or cross product of two vectors is defined as a vector having a magnitude equal to the product of the magnitudes of two vectors with the sine of angle between them and direction perpendicular to the plane containing the two vectors in accordance with **right hand screw rule**.

Thus if $\vec{A}$ & $\vec{B}$ are two vectors, then their vector product written as $\vec{A} \times \vec{B}$ is a vector $\vec{C}$ defined by $\vec{C} = \vec{A} \times \vec{B} = AB \sin\theta\ \hat{n}$

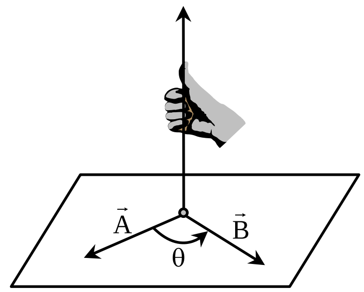
Where A and B are the magnitudes of $\vec{A}$ and $\vec{B}$ respectively and θ is the smaller angle between the two. Where $\hat{n}$ is the unit vector whose direction is perpendicular to the plane containing the two vectors, in accordance with right hand screw rule.

6.2.1 Right hand screw rule :

(a) A right hand screw whose axis is perpendicular to the plane framed by $\vec{A}$ and $\vec{B}$ is rotated from $\vec{A}$ to $\vec{B}$ through the smaller angle between them, then the direction of advancement of the screw gives the direction of $\vec{A} \times \vec{B}$ i.e. $\vec{C}$



(b) Place the vector $\vec{A}$ and $\vec{B}$ tail to tail (this defines a plane). Now place stretched fingers and thumb of right hand perpendicular to the plane of $\vec{A}$ and $\vec{B}$ such that the fingers are along the vectors $\vec{A}$.If the fingers are now closed through smaller angle so as to go towards $\vec{B}$. The thumb gives the direction of $\vec{A} \times \vec{B}$ i.e. $\vec{C}$.



(c) Cross product non-commutative :

$\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$

(d) Follows distributive law :

$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

(e) Does not follows associative law :

$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$

(f) $\hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}$

and $\hat{i} \times \hat{i} = 0, \hat{j} \times \hat{j} = 0, \hat{k} \times \hat{k} = 0$

(g) $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$

$\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$

$\vec{a} \times \vec{b} = (a_x\hat{i} + a_y\hat{j} + a_z\hat{k})$

$\times\ (b_x\hat{i} + b_y\hat{j} + b_z\hat{k})$

$= (a_y b_z - a_z b_y)\ \hat{i} + (a_z b_x - a_x b_z)\ \hat{j}$

$+ (a_x b_y - a_y b_x)\hat{k}$

6.2.2 Properties of vector (or cross product)

(i) If **a** and **b** are any vectors, and m is any real number (positive or negative) then

(m **a**) × **b** = m (**a** × **b**) = **a** × (m **b**)

(ii) The vector product is distributive w.r.t. addition

a × (**b** + **c**) = **a** × **b** + **a** × **c**

(iii) Vector product of two vectors in terms of their rectangular components

$\mathbf{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\mathbf{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\mathbf{a}\times\mathbf{b} = (a_2b_3 - a_3b_2)\,\hat{i} - (a_1b_3 - a_3b_1)\,\hat{j} + (a_1b_2 - a_2b_1)\,\hat{k}$$

(iv) Angle between two vectors

$\mathbf{a}\times\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\sin\theta$

$\Rightarrow$ $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\sin\theta$

$\sin\theta = |\mathbf{a}\times\mathbf{b}| / |\mathbf{a}|\,|\mathbf{b}|$

Let

$\mathbf{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\mathbf{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$$\mathbf{a}\times\mathbf{b} = (a_2b_3 - a_3b_2)\,\hat{i} - (a_1b_3 - a_3b_1)\,\hat{j} + (a_1b_2 - a_2b_1)\,\hat{k}$$

$\Rightarrow |\mathbf{a}\times\mathbf{b}| =$

$$\sqrt{(a_2b_3 - a_3b_2)^2 + (a_1b_3 - a_3b_1)^2 + (a_1b_2 - a_2b_1)^2}$$

$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, $|\mathbf{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$

$\sin\theta =$

$$\frac{\sqrt{(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2}}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

6.2.3 Unit vector perpendicular to two given vectors

Let $\hat{n}$ be a unit vector perpendicular to two (non-zero) vectors **a**, **b** and positive for right handed rotation from **a** to **b** and θ be the angle between the vectors **a**, **b** then

$\mathbf{a}\times\mathbf{b} = ab\sin\theta\ \hat{n}$

$|\mathbf{a}\times\mathbf{b}| = ab\sin\theta$

Thus we get $= \mathbf{a}\times\mathbf{b} / |\mathbf{a}\times\mathbf{b}|$

Example based on Vector Product

Ex.21 Prove that

$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) + \mathbf{b}\times(\mathbf{c}+\mathbf{a}) + \mathbf{c}\times(\mathbf{a}+\mathbf{b}) = 0$

Sol. $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) + \mathbf{b}\times(\mathbf{c}+\mathbf{a}) + \mathbf{c}\times(\mathbf{a}+\mathbf{b})$

$= \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c} + \mathbf{b}\times\mathbf{c} + \mathbf{b}\times\mathbf{a} + \mathbf{c}\times\mathbf{a} + \mathbf{c}\times\mathbf{b}$

$= \mathbf{a}\times\mathbf{b} - \mathbf{c}\times\mathbf{a} + \mathbf{b}\times\mathbf{c} - \mathbf{a}\times\mathbf{b} + \mathbf{c}\times\mathbf{a} - \mathbf{b}\times\mathbf{c} = 0$

Ex.22 Find **a** × **b** and **b** × **a** if

(i) $\mathbf{a} = 3\hat{k} + 4\hat{j}$, $\mathbf{b} = \hat{i} + \hat{j} - \hat{k}$

(ii) $\mathbf{a} = (2, -1, 1)$; $\mathbf{b}\ (3, 4, -1)$

Sol. (i) $$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 4 & 3 \\ 1 & 1 & -1 \end{vmatrix} = -7\hat{i} + 3\hat{j} - 4\hat{k}$$

$$\mathbf{b}\times\mathbf{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ 0 & 4 & 3 \end{vmatrix} = 7\hat{i} - 3\hat{j} + 4\hat{k}$$

(ii) $$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix} = -3\hat{i} + 5\hat{j} + 11\hat{k}$$

$$\mathbf{b}\times\mathbf{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & -1 \\ 2 & -1 & 1 \end{vmatrix} = 3\hat{i} - 5\hat{j} - 11\hat{k}$$

Ex.23 If $\mathbf{a} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\mathbf{b} = 2\hat{i} - 2\hat{j} + 4\hat{k}$

(i) find the magnitude of **a** × **b**

(ii) find a unit vector perpendicular to both **a** and **b**.

(iii) find the cosine and sine of the angle between the vectors **a** and **b**

Sol. (i) $$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 2 \\ 2 & -2 & 4 \end{vmatrix} = 8\hat{i} - 8\hat{j} - 8\hat{k}$$

∴ Magnitude of $\mathbf{a}\times\mathbf{b} = |\mathbf{a}\times\mathbf{b}|$

$$= \sqrt{(8)^2 + (-8)^2 + (-8)^2} = 8\sqrt{3}$$

(ii) $$\hat{n} = \frac{a\times b}{|a\times b|} = \frac{8\hat{i} - 8\hat{j} - 8\hat{k}}{8\sqrt{3}} = \frac{1}{\sqrt{3}}(\hat{i} - \hat{j} - \hat{k})$$

There are two unit vectors perpendicular to both a and b they are

$$\pm\ \hat{n} = \pm\ \frac{1}{\sqrt{3}}(\hat{i} - \hat{j} - \hat{k})$$

(iii) To find cos θ

$\mathbf{a}.\mathbf{b} = ab\cos\theta = (3\hat{i} + \hat{j} + 2\hat{k}).(2\hat{i} - 2\hat{j} + 4\hat{k})$

$= (3)(2) + (1)(-2) + 2(4) = 12$

$a = |\mathbf{a}| = \sqrt{3^2 + 1^2 + 2^2} = \sqrt{14}$

$b = |\mathbf{b}| = \sqrt{(2)^2 + (-2)^2 + (4)^2} = \sqrt{24}$

$$\cos\theta = \mathbf{a}.\mathbf{b}/ab = \frac{12}{\sqrt{14}\sqrt{24}} = \frac{12}{\sqrt{2}\sqrt{7}.2\sqrt{2}\sqrt{3}} = \sqrt{\frac{3}{7}}$$

$$\text{Also, } \sin\theta = |\mathbf{a}\times\mathbf{b}|/\ ab = \frac{8\sqrt{3}}{\sqrt{14}\sqrt{24}} = \frac{2}{\sqrt{7}}$$

Also, $\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{3}{7}} = \sqrt{\frac{4}{7}} = \frac{2}{\sqrt{7}}$

Ex.24 The vectors from origin to the points A and B are $\mathbf{a} = 3\hat{i}-6\hat{j}+2\hat{k}$ and $\mathbf{b} = 2\hat{i}+\hat{j}-2\hat{k}$ respectively. Find the area of :

(i) the triangle OAB

(ii) the parallelogram formed by **OA** and **OB** as adjacent sides.

Sol. Given **OA** = **a** = $3\hat{i} - 6\hat{j} + 2\hat{k}$

and **OB** = **b** = $2\hat{i} + \hat{j} - 2\hat{k}$

$$\therefore (\mathbf{a}\times\mathbf{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -6 & 2 \\ 2 & 1 & -2 \end{vmatrix}$$

$= (12 - 2)\hat{i} - (-6 - 4)\hat{j} + (3 + 12)\hat{k}$

$= 10\hat{i} + 10\hat{j} + 15\hat{k}$

$\Rightarrow |\mathbf{a}\times\mathbf{b}| = \sqrt{10^2+10^2+15^2} = \sqrt{425} = 5\sqrt{17}$

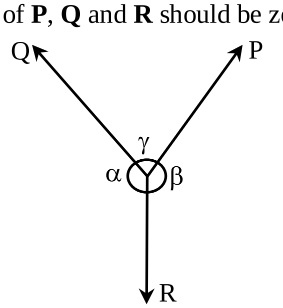
(i) Area of ΔOAB = $\frac{1}{2}|\mathbf{a}\times\mathbf{b}| = \frac{1}{2} . 5\sqrt{17}$ sq. units

$= \frac{5}{2}\sqrt{17}$ sq. units

(ii) Area of parallelogram formed by **OA** and **OB** as adjacent sides = $|\mathbf{a}\times\mathbf{b}| = 5\sqrt{17}$ sq. units.

7. LAMI'S THEOREM

If three forces **P**, **Q** and **R** are acting at a point then, if the point is in equilibrium then, the vector summation of **P**, **Q** and **R** should be zero. Then



$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma}$$

SOLVED EXAMPLES

Ex.1 A vector **a** is turned without a changed in its length through a small angle $d\theta$. What are $|\Delta\mathbf{a}|$ and Δa.

Sol. $|\Delta\mathbf{a}| = a d\theta$

$\Delta a = 0$

Ex.2 The magnitude of a vector **A** is 10 units and it makes an angle of 30º with the X-axis. Find the components of the vector if it lies in the X-Y plane.

Sol. Components of vector **A** lying in the X-Y plane are

$A_x = A\cos\theta$, $A_y = A\sin\theta$, $A_z = 0$

Thus $A_x = 10\cos 30^\circ = \frac{10\sqrt{3}}{2}$

= 8.66 units

$A_y = 10\sin 30^\circ = 10 \times 1/2$

= 5 units

$A_z = 0$

Ex.3 Let $\mathbf{A} = 2\hat{i} + \hat{j}$, $\mathbf{B} = 3\hat{j} - \hat{k}$ and $C = 6\hat{i} - 2\hat{k}$. Find (i) **A** + **B** + **C** (ii) 2**A** – 3**B** and (iii) **A** – **B** – **C**

Sol. From the given data, we find that (check yourself)

(i) **A** + **B** + **C** = $8\hat{i} + 4\hat{j} - 3\hat{k}$

(ii) 2**A** – 3**B** = $4\hat{i} - 7\hat{j} + 3\hat{k}$

(iii) **A** – **B** – **C** = $-4\hat{i} - 2\hat{j} + 3\hat{k}$

Ex.4 Two forces F_1 = 1N and F_2 = 2N act along the lines x = 0 and y = 0 respectively. Then the resultant force would be –

Sol. x = 0 means y – axis

y = 0 means x – axis

$\therefore$ 1N is acting along y-axis and 2N is acting along x -axis

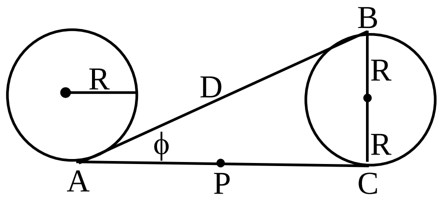
F = $2\hat{i} + \hat{j}$

Ex.5 What is the displacement of the point of the wheel initially in contact with the ground when the wheel roles forward half a revolution ? Take the radius of the wheel R and x-axis in forward direction.

(A) R (B) $R \times \sqrt{\pi^2+4}$

(C) 2πR (D) πR

Sol. In accordance with fig during the half revolution of the wheel, the point A covers πR = (AC) horizontal distance while 2R (= BC) vertical distance,



So here P = πR ; θ = 2R and π = 90º

So D = $\sqrt{(\pi R)^2 + (2R)^2}$ = R $\sqrt{\pi^2 + 4}$

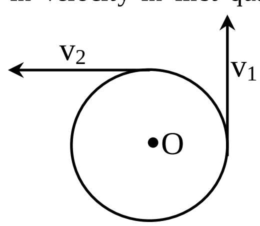
and $\phi = \tan^{-1}\left[\frac{2R}{\pi R}\right] = \tan^{-1}\left[\frac{2}{\pi}\right]$

i.e. displacement has magnitude R $\sqrt{\pi^2 + 4}$

and makes an angle $\tan^{-1}\left(\frac{2}{\pi}\right)$ with x-axis.

Hence correct answer is (B).

Ex.6 A body is moving with uniform speed v on a horizontal circle in anticlockwise direction starting from A as shown in figure. What is the change in velocity in first quarter revolution.



(A) $\sqrt{2}$ v south-west (B) v south-north

(C) $\frac{v}{\sqrt{2}}$ south-west (D) $\sqrt{2}$ v south-east

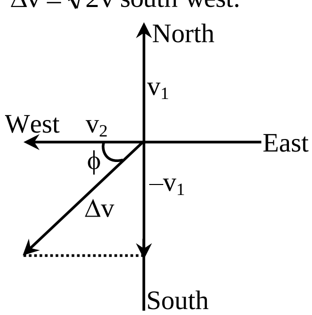
Sol. As shown in fig for quarter revolution

$\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$ and θ = 90º,

So $\Delta\vec{v} = \sqrt{v^2 + v^2} = (\sqrt{2})v$

$\phi = \tan^{-}\left(\frac{v}{v}\right)$ = 45º

$\Delta\vec{v} = \sqrt{2}v$ south west.



Hence correct answer is (A).

Ex.7 The sum of the magnitudes of two forces acting at a point is 18 and the magnitude of their resultant is 12. If the resultant is at 90º with the force of smaller magnitude, what are the magnitudes of forces ?

(A) 12, 5 (B) 14, 4

(C) 5, 13 (D) 10, 8

Sol. Let P be the smaller force then it is given that

P + Q = 18 ..... (1)

R = $\sqrt{P^2 + Q^2 + 2PQ\cos\theta}$ = 12 ..... (2)

Q sin θ / P + Q cos θ = tan ϕ = tan 90º = ∞

P + Q cos θ = 0 ..... (3)

Substituting the value of P

Q (1 – cos θ) = 18 .....(4)

and subtracting square of equation (2) from (1)

2PQ [1 – cos θ] = $18^2 - 12^2$ = 180 .... (5)

Dividing equation (5) by (4)

2P = 10 i.e. P = 5, So Q = 13

So the magnitude of forces are (5 and 13)

Hence correct answer is (C).

Ex.8 What vector must be added to the vector $\hat{i} - 3\hat{j} + 2\hat{k}$ and $3\hat{i} + 6\hat{j} - 7\hat{k}$ so that the resultant vector is a unit vector along the y-axis ?

Sol. Let the vector added to the summation of $(\hat{i} - 3\hat{j} + 2\hat{k}) + (3\hat{i} + 6\hat{j} - 7\hat{k})$ be $x\hat{i} + y\hat{j} + z\hat{k}$

$\therefore$ $(4\hat{i} + 3\hat{j} - 5\hat{k}) + (x\hat{i} + y\hat{j} + z\hat{k}) = \hat{j}$

(4 + x) $\hat{i}$ + (3 + y) $\hat{j}$ + (–5 + z) $\hat{k}$ = $\hat{j}$

Comparing both the sides

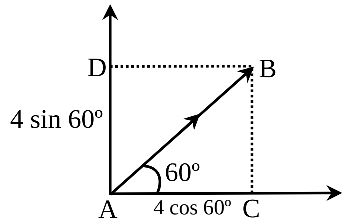
4 + x = 0 ; 3 + y = 1 ; –5 + z = 0

x = – 4 ; y = 1 – 3 = –2 ; z = 5

$\therefore$ vector is : – 4 $\hat{i}$ – 2 $\hat{j}$ + 5 $\hat{k}$

Ex.9 Let **AB** be a vector in two dimensional plane with magnitude 4 units, and making an angle of 60º with x-axis and lying in first quadrant. Find the components of AB along x-axis and y-axis. Hence represent **AB** in terms of unit vectors $\hat{i}$ and $\hat{j}$. Also verify that calculation of components is correct .

Sol. Taken A is origin from the figure, we see that component on x- axis is AC



$= 4\cos 60^\circ = 4\,\frac{1}{2} = 2$ and

Component along y-axis is AD

$= 4\sin 60^\circ = 4\frac{\sqrt{3}}{2} = 2\sqrt{3}$

Hence **AB** = $2\hat{i} + 2\sqrt{3}\,\hat{j}$

verification : $\sqrt{(2)^2 + (2\sqrt{3})^2}$

$= 4$

Ex.10 If $\mathbf{A} = 3\hat{i} + 4\hat{j}$ and $\mathbf{B} = 7\hat{i} + 24\hat{j}$, the vector having the same magnitude as **B** and parallel to **A** is –

Sol. Let the required vector be

$\mathbf{C} = C_x\,\hat{i} + C_y\,\hat{j}$

Given that C is parallel to **A**, then

$$\frac{C_y}{C_x} = \frac{4}{3} \qquad \text{...... (i)}$$

and |**C**| is equal to |**B**|, thus

$\sqrt{C_x^2 + C_y^2} = \sqrt{7^2 + (24)^2} = \sqrt{49 + 576}$

or $\sqrt{C_x^2 + C_y^2} = \sqrt{625}$

$$C_x^{\,2} + C_y^{\,2} = 625 \qquad \text{......(ii)}$$

Solving equation (i) and (ii) we get

$C_x = 15$ and $C_y = 20$

$\mathbf{C} = 15\hat{i} + 20\hat{j}$

Alter method :

$|\mathbf{B}| = \sqrt{7^2 + (24)^2} = \sqrt{625} = 25$

unit vector in the direction of **A**

$\hat{A} = \frac{3\hat{i} + 4\hat{j}}{5}$

Required vector $= 25\left(\frac{3\hat{i} + 4\hat{j}}{5}\right) = 15\hat{i} + 20\hat{j}$

Ex.11 Two vectors

$\mathbf{A} = 2\hat{i} + 2\hat{j} + p\hat{k}$ and $\mathbf{B} = \hat{i} + \hat{j} + \hat{k}$

are given. Find the value of p if

(i) the two vectors are perpendicular

(ii) the two vectors are parallel

Sol. (i) **A** and **B** will be perpendicular if **A**.**B** = 0

Now **A . B** = $A_xB_x + A_yB_y + A_zB_z$

$= 2 \times 1 + 2 \times 1 + p \times 1$

$= 4 + p$

for **A . B** = 0, we must have

$0 = 4 + p$

or $p = -4$

(ii) **A** and **B** will be parallel if **A** × **B** = 0

Now $\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & p \\ 1 & 1 & 1 \end{vmatrix}$

$= \hat{i}\,(2 - p) + \hat{j}\,(p - 2) + \hat{k}\,(2 - 2)$

$= \hat{i}\,(2 - p) + \hat{j}\,(p - 2)$

For **A** × **B** = 0, we must have each component to be zero. That is 0 = 2 – p , and 0 = p – 2 (both conditions similar). Thus p = 2

Ex.12 Let for two vector **A** and **B**, sum (**A** + **B**) is perpendicular to the difference (**A** – **B**). Find the ratio A/B of their amplitudes.

Sol. It is given that (**A** + **B**) is perpendicular to (**A** – **B**). Thus

(**A** + **B**). (**A** – **B**) = 0

or $\mathbf{A}^2 + \mathbf{B}\,.\,\mathbf{A} - \mathbf{A}\,.\,\mathbf{B} - B^2 = 0$

Because of commutative property of dot product

A . B = B . A

∴ $A^2 - B^2 = 0$

or A = B

Thus A/B = 1

i.e. the ratio of magnitudes is 1.

Ex.13 Find out the angle between

$A = 3\hat{i} + 2\hat{j} + \hat{k}$

$B = 5\hat{i} - 2\hat{j} - 3\hat{k}$

Sol. $|A| = \sqrt{3^2 + 2^2 + 1^2} = \sqrt{14}$

$|B| = \sqrt{(5)^2 + (-2)^2 + (-3)^2} = \sqrt{38}$

A . **B** = $A_xB_x + A_yB_y + A_zB_z$

$= 3 \times 5 + 2(-2) + (1)(-3) = 8$

$\cos\theta = \frac{A.B}{|A||B|} = \frac{8}{\sqrt{14 \times 38}}$

Ex.14 Find a unit vector perpendicular to both $\mathbf{A} = 2\hat{i} + \hat{j}$ and $\mathbf{B} = \hat{i} + 2\hat{j}$

Sol. A unit vector $\hat{n}$ which is perpendicular to both A and B is obtained from the relation-

$$\hat{n} = \frac{A \times B}{|A \times B|}$$

Now $\mathbf{A} \times \mathbf{B} = (2\hat{i} + \hat{j}) \times (\hat{i} + 2\hat{j})$

$= -\hat{k} + 4\hat{k} = 3\hat{k}$

and |**A** × **B**| = 3

Therefore $\hat{n} = \frac{3\hat{k}}{3} = \hat{k}$

The unit vector $\hat{k}$ is perpendicular to both $2\hat{i} + \hat{j}$ and $\hat{i} + 2\hat{j}$.

Ex.15 If the sum of two unit vectors is a unit vector, then magnitude of difference is –

(A) $\sqrt{2}$ (B) $\sqrt{3}$

(C) $1/\sqrt{2}$ (D) $\sqrt{5}$

Sol.(B)

Let $\hat{n}_1$ and $\hat{n}_2$ are the two unit vectors, then the sum is

$$\mathbf{n}_s = \hat{n}_1 + \hat{n}_2$$

or $n_s^2 = n_1^2 + n_2^2 + 2n_1n_2 \cos\theta$

$= 1 + 1 + 2\cos\theta$

since it is given that n_s is also a unit vector, therefore

$$1 = 1 + 1 + 2\cos\theta$$

or $\cos\theta = -\frac{1}{2}$ or $\theta = 120^{o}$

Now the difference vector is

$$\mathbf{n}_d = \mathbf{n}_1 - \mathbf{n}_2$$

or $n_d^2 = n_1^2 + n_2^2 - 2n_1n_2\cos\theta$

$= 1 + 1 - 2\cos(120^{o})$

$= 2 - 2(-1/2) = 2 + 1 = 3$

$\therefore$ $n_d = \sqrt{3}$

Thus the correct answer is (B)

Ex.16 The torque of a force $\mathbf{F} = -3\hat{i} + \hat{j} + 5\hat{k}$ acting at the point $\mathbf{r} = 7\hat{i} + 3\hat{j} + \hat{k}$ is –

(A) $14\hat{i} - 38\hat{j} + 16\hat{k}$ (B) $4\hat{i} + 4\hat{j} + 6\hat{k}$

(C) $-21\hat{i} + 4\hat{j} + 4\hat{k}$ (D) $-14\hat{i} + 34\hat{j} - 16\hat{k}$

Sol.(A) The torque is defined as $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$

$$\mathbf{r} \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & 3 & 1 \\ -3 & 1 & 5 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} 3 & 1 \\ 1 & 5 \end{vmatrix} + \hat{j}\begin{vmatrix} 1 & 7 \\ 5 & -3 \end{vmatrix} + \hat{k}\begin{vmatrix} 7 & 3 \\ -3 & 1 \end{vmatrix}$$

$= \hat{i}(15 - 1) + \hat{j}(-3 - 35) + \hat{k}(7 - (-9))$

$= 14\hat{i} - 38\hat{j} + 16\hat{k}$

Thus the answer is (A)

Ex.17 If $\mathbf{A} \times \mathbf{B} = \mathbf{C}$, then which of the following statements is wrong –

(A) $\mathbf{C} \perp \mathbf{A}$ (B) $\mathbf{C} \perp \mathbf{B}$

(C) $\mathbf{C} \perp (\mathbf{A} + \mathbf{B})$ (D) $\mathbf{C} \perp (\mathbf{A} \times \mathbf{B})$

Sol.(D) From the property of vector product, we notice that **C** must be perpendicular to the plane formed by vector **A** and **B**. Thus **C** must be perpendicular to **A** and to **B**. Thus the statements (1) and (2) are perpendicular correct. Now **A** + **B** vector also must lie in the plane formed by vector **A** and **B**. Thus **C** must be perpendicular to **A** + **B** also The cross product (**A** × **B**) gives a vector **C** which can not be perpendicular to itself. Thus the statement (D) is wrong.